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ON A CONSTRUCTION OF INFINITE FAMILIES OF REGULAR CAYLEY MAPS

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Regular Cayley maps are Cayley maps possessing the highest possible level of symmetry. In our paper, we introduce a construction of finite groups from a given set of generators, that provides us with infinite families of regular Cayley maps.

1. Introduction

Among all the Cayley maps the regular Cayley maps are those which cause the greatest interest. In the last twenty years many questions have arised concerning these highly symmetrical embeddings of Cayley graphs. One of the best known problems in this area is the characterization of Cayley graphs which admit a regular 2-cell embedding as a Cayley map on an orientable surface. Using the notation introduced later in the paper, we can simply state this problem as a characterization of those of finite groups G with suitable sets of generators Ω and involutions $\chi(y)=y^{-1}$, $y\in\Omega$, which admit an existence of a cyclic permutation p on Ω such that the Cayley map $CM(G,\Omega,\chi,p)$ is regular (see [1], [2]).

In our paper we solve a problem, in some sense, opposite to the problem just stated. Given a set of generators Ω, an involution χ and a cyclic permutation p on Ω, is there a finite group G such that the Cayley map $CM(G,\Omega,\chi,p)$ is regular? In the following, we provide a positive answer to this question for two types of permutations p, namely for balanced and antibalanced permutations ([1], [8]). Our proof is based on a construction of finite groups from their generators and a cyclic permutation on these generators. In special cases when the permutation used for the construction is balanced or antibalanced the obtained groups underly regular Cayley maps, what enables us to construct infinite families of those. At the end of our paper, we also briefly discuss the existence of regular Cayley maps that are neither balanced nor antibalanced.

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2. Preliminaries.

Having a finite group G and a set Ω of generators of G not containing the unity e but containing x^{-1} together with every x contained in Ω, we define a *Cayley graph* $K = C(G,\Omega)$ as follows. The vertex set of K is the set of elements of G where the arc set of K is the set $G\times\Omega$, and the incidence relation is defined by letting an arc $(g,x) \in G\times\Omega$ have initial vertex g and terminal vertex $g\cdot x$. A *Cayley map* $M = CM(G,\Omega,\chi,p)$ is a 2-cell embedding of Cayley graph $C(G,\Omega)$ on an orientable surface, with the same local orientation p at every vertex (p being a cyclic permutation on Ω). Here χ denotes the inverse involution on Ω associating element $y\in\Omega$ with its inverse y^{-1}. Since χ is uniquely determined given G and Ω, it is usual to abbreviate $CM(G,\Omega,\chi,p)$ as $CM(G,\Omega,p)$. The triple (Ω,χ,p) is called the *type* of the Cayley map. Obviously, the notion of a type (Ω,χ,p) can be defined for an arbitrary nonempty finite set Ω, involution χ and cyclic permutation p, both acting on Ω, and can be used without any reference to a group.

If p, in addition, satisfies the property $p(x^{-1}) = (p(x))^{-1}$, for all $x\in\Omega$, then p is called *balanced* and the map is called a *balanced Cayley map*. In the case when p satisfies $p(x^{-1}) = (p^{-1}(x))^{-1}$, for all $x\in\Omega$, both the permutation and the map are called *antibalanced*. The balanced Cayley maps are members of a bigger family of k-*balanced* Cayley maps, for which $p^k(x^{-1}) = (p^k(x))^{-1}$, for all $x\in\Omega$ and integral k ([3]).

Further we associate with every Cayley map $CM(G,\Omega,\chi,p)$ two permutations on the set of arcs: the rotation P and the arc-reversing involution T. We define them, for every arc $(g,x)\in G\times\Omega$, by $P(g,x) = (g,p(x))$, and $T(g,x) = (gx,x^{-1})$.

A *map-automorphism* A of a Cayley map $CM(G,\Omega,\chi,p)$ is a permutation on the set of arcs satisfying the conditions $AP = PA$ and $AT = TA$. The set of all map-automorphisms of M together with the operation of composition gives rise to a group, denoted by $\mathrm{Aut}\,M$. It can be easily shown that for any two arcs (g,x) and (f,y) there is at most one map automorphism taking (g,x) to (f,y) ([1]). It follows that $|\mathrm{Aut}\,M| \le |G|\cdot|\Omega|$. If the group $\mathrm{Aut}\,M$ acts transitively on the set of arcs of M (which is equivalent to the identity $|\mathrm{Aut}\,M| = |G|\cdot|\Omega|$), we say that M is a *regular map*.

In [3] one can find a characterization of regular Cayley maps using the notion of a rotary mapping. Let us briefly introduce the basic definitions and facts about rotary mappings and their relation to regular Cayley maps.

A bijection ρ of the group G onto itself is called a *rotary mapping* of the Cayley map $M = CM(G,\Omega,\chi,p)$ if, for all $a\in G$ and $x\in\Omega$, ρ satisfies the following properties:

- $\rho(e) = e$
- $\rho(a)^{-1}\cdot\rho(ax)\in\Omega$
- $\rho(a)^{-1}\rho(a\cdot p(x)) = p(\rho(a)^{-1}\rho(ax))$,

(where the first and second condition simply require ρ to be a graph automorphism of $C(G,\Omega)$ that preserves the identity.)

The following theorem characterizes the regular Cayley maps. (For the proof see [3].)

Theorem 1. *Let $M = CM(G,\Omega,\chi,p)$ be a Cayley map. If there exists a rotary mapping ρ of M whose restriction to Ω is equal to p ($\rho|\Omega = p$) then M is regular. Conversely, if M is a regular Cayley map then such a rotary mapping exists.*

If the rotary mapping ρ, which gives rise to the regularity of M, exists, it satisfies, for every $x_1,\dots,x_n \in \Omega$, the following formula:

$$\rho(x_1 \dots x_n) = b_1 \dots b_n, \tag{1}$$

where $b_1 = p(x_1)$, $b_{i+1} = p^{l_i}(b_i^{-1})$ and l_i are determined by the equations $x_{i+1} = p^{l_i}(x_i^{-1}), 1 \le i \le n-1$.

3. The construction of regular Cayley maps for a given type (Ω,χ,p).

In the following we construct a group G for which $CM(G,\Omega,\chi,p)$ is regular, for a given generating set Ω, an involution χ and a cyclic balanced or antibalanced permutation p on Ω.

Let Ω be an arbitrary finite set and let χ be an involution on Ω. We will call the pair (Ω,χ) a *generating set* and the element $\chi(y)$ the *inverse of* y, $y \in \Omega$. Let $F(\Omega,\chi)$ be the factorization of the free group on Ω by the set of equations $\{y\chi(y) = \chi(y)y = e | y \in \Omega\}$, i. e. $F(\Omega,\chi)$ be the infinite group generated by Ω in which $\chi(y)$ becomes a (group) inverse of y, for every $y \in \Omega$. Now let G be an arbitrary finite group. We say that the generating set (Ω,χ) is a *proper generating set for* G provided G is a factorization of $F(\Omega,\chi)$ satisfying the property $x \neq y$ in G for each pair of distinct elements x,y from Ω. (It is important to notice that not every generating set (Ω,χ) generating G is a *proper* generating set for G, as an identity of the type $x = y$, $x,y \in \Omega$, can easily force an "identification" of two originally distinct elements of Ω. We shall encounter this kind of a situation quite often.) Obviously now, everything has the usual meaning: whenever (Ω,χ) is a proper generating set for G, Ω is a generating set for G containing y^{-1} together with every y but not containing e.

Using the introduced notions we can proceed to the very construction of finite groups for a given type (Ω,χ,p).

Let (Ω,χ) be a generating set, p be a cyclic permutation on Ω and let $F(\Omega,\chi)$ be the above defined infinite group. For $n \ge 2$, define $F_n(\Omega,\chi)$ to be the set of all irreducible words in $F(\Omega,\chi)$ of the length n (it is the set of all words $x_1 \dots x_n$ in the free group over Ω, where $x_i \neq \chi(x_{i+1})$, for all $1 \le i \le n-1$). Since $F(\Omega,\chi)$ (as a set) is a union of all $F_n(\Omega,\chi)$, we can define a mapping ρ from $F(\Omega,\chi) - \{e\}$ to $F(\Omega,\chi)$ by means of formula (1) (where the element e is the unit of $F(\Omega,\chi)$). Extending the definition of ρ by setting $\rho(e)=e$ we get a mapping from $F(\Omega,\chi)$ to $F(\Omega,\chi)$. We claim that ρ is a bijection.

First notice that ρ preserves Ω. Then let us prove that ρ preserves the sets $F_n(\Omega,\chi)$ and sends irreducible words of length $n\geq 2$ onto irreducible words of the same length. Since, by the definition, ρ sends the words of length n to words of length n, it suffices to prove that $\rho(x_1\dots x_n)$ is irreducible whenever $x_1\dots x_n$ is irreducible. Let $\rho(x_1\dots x_n)=b_1\dots b_n$ and suppose, on the contrary, that $b_1\dots b_n$ is not χ-irreducible. Then $b_{i+1}=\chi(b_i)=b_i^{-1}$, for some $1\leq i\leq n-1$. At the same time one can see from formula (1) that $b_{i+1}=p^{l_i}(b_i^{-1})$ and therefore, combining with the previous, we get $b_i^{-1}=p^{l_i}(b_i^{-1})$. Thus, $l_i=0$ and $x_{i+1}=p^0(x_i^{-1})=x_i^{-1}$, which contradicts the irreducibility of $x_1\dots x_n$ and proves that ρ sends irreducible words to irreducible words. Further, since all the $F_n(\Omega,\chi)$ are finite and ρ is apparently injective, ρ is bijective on the $F_n(\Omega,\chi)$'s and therefore bijective.

Using the just defined bijection, for any $n\geq 2$, define the group $G_n(\Omega,\chi,p)$ to be the factorization of $F(\Omega,\chi)$ by the set of identities

$$\rho(x_1\dots x_n)=\rho^2(x_1\dots x_n)=\dots=\rho^{|\Omega|}(x_1\dots x_n), \tag{2}$$

where $x_1\dots x_n$ ranges through the whole set $F_n(\Omega,\chi)$. (Notice here that by the definition of ρ the last member of the series of equations is actually equal to $x_1\dots x_n$.)

Theorem 2. *Let (Ω,χ) be a generating set and let p be an arbitrary cyclic permutation on Ω. Then, for each $n\geq 2$, the group $G_n(\Omega,\chi,p)$ is a finite group.*

Proof. Let $n\geq 2$ be fixed. We will prove that every word from $F_m(\Omega,\chi)$, $m>n$, is equal to a word of length n or less, i. e. that $G_n(\Omega,\chi,p)$ has only finitely many elements. Let $x_1\dots x_m\in F_m(\Omega,\chi)$ with $m>n$. The first n letters of $x_1\dots x_m$ belong to $F_n(\Omega,\chi)$ and therefore:

$$x_1\dots x_m=\rho(x_1\dots x_n)\cdot x_{n+1}\dots x_m=\dots=\rho^{|\Omega|-1}(x_1\dots x_n)\cdot x_{n+1}\dots x_m.$$

Repeated application of formula (1) reveals the following useful observation. The powers of the mapping ρ satisfy the formula $\rho^i(x_1\dots x_n)=c_1\dots c_n$, where $c_1=p^i(x_1)$, and the powers l_j are determined by the original equations $x_{j+1}=p^{l_j}(x_j^{-1})$. Therefore all the powers of ρ have the same system of exponents. As i ranges from 1 through $|\Omega|$, the first letter of $\rho^i(x_1\dots x_n)$ ranges through all the elements of Ω, and consequently, all the letters of $\rho^i(x_1\dots x_n)$ range through all Ω. In particular, exactly one of the words

$$x_1\dots x_n,\rho(x_1\dots x_n),\dots,\rho^{|\Omega|-1}(x_1\dots x_n)$$

ends with the letter x_{n+1}^{-1}. Therefore $x_1\dots x_m$ is equal to a word of length $m-2$, obtained by canceling the two subsequent letters $x_{n+1}^{-1}x_{n+1}$. Thus every word of length $m>n$ is equal to a shorter word and proceeding inductively we can conclude that every word of length $m>n$ is equal to a word of length n or less (it is actually n or $n-1$, for reduced words). It follows that $G_n(\Omega,\chi,p)$ is finite for every $n\geq 2$. ∎

The introduced construction does not put any specific limits on the permutation p, except of the fact that its action on Ω has to be cyclic (this condition is essential for the proof of finiteness of the groups G_n). It is therefore reasonable to think about a Cayley map $CM(G_n(\Omega,\chi,p),\Omega,p)$ for any generating set (Ω,χ) and cyclic p. One might be very easily tending to think that all these are regular Cayley maps, and, in fact, our original intention was to construct in this manner regular Cayley maps with arbitrary permutations p. There are, however, two principal limitations to this approach.

First of all, although the factorization using identities (2) is effecting the words of length $n\geq 2$ primarily, there is no guarantee that identities (2) will not imply a consequence of the type $x=y$ for x and y in Ω. In other words, there is no guarantee for (Ω,χ) to be a *proper* generating set for $G_n(\Omega,\chi,p)$. In fact, as we shall see in Example 3. at the end of this article, in some cases the identities (2) force $G_n(\Omega,\chi,p)$ to collapse into a trivial one- or two-element group (i. e. all the elements of $F(\Omega,\chi)$ get eventually identified).

Second, even if $G_n(\Omega,\chi,p)$ happens to be generated by (Ω,χ) in the "right way", the mapping ρ defined by formula (1) does not necessarily have to be a rotary mapping for $CM(G_n(\Omega,\chi,p),\Omega,p)$. Without some specific conditions on the permutation p, we are always in danger that the mapping ρ is not "well-defined". Namely, for two different expressions of an element a, in terms of elements of Ω, $x_1\dots x_k=a=y_1\dots y_m$, it is not necessarily true that $\rho(x_1\dots x_k)=\rho(y_1\dots y_m)$.

Despite these two drawbacks of the introduced method, it is powerful enough to produce infinite families of regular Cayley maps. As we shall see in the following theorem and examples, there are two types of permutations that guarantee the Cayley map $CM(G_n(\Omega,\chi,p),\Omega,p)$ to be regular whenever (Ω,χ) happens to be a proper generating set for $G_n(\Omega,\chi,p)$.

Theorem 3. *Let (Ω,χ) be a generating set, p be a balanced or antibalanced cyclic permutation on Ω, $n\geq 2$, and suppose that (Ω,χ) is a proper generating set for $G_n(\Omega,\chi,p)$. Then the Cayley map $CM(G_n(\Omega,\chi,p),\Omega,p)$ is regular.*

Proof. For the special case of balanced or antibalanced p, the formula (1) achieves a much simpler form. Suppose first p to be balanced. We get easily by induction that $p^k(x^{-1})=(p^k(x))^{-1}$, for all $x\in\Omega$ and integral k. We claim that

$$\rho(x_1\dots x_m)=p(x_1)p(x_2)\dots p(x_m). \tag{3}$$

To prove the formula (3) we proceed by induction on m. If $m=1$ then the formula (1) yields $\rho(x_1)=p(x_1)$. Suppose that (3) is true for $m-1$. Then

$$\rho(x_1\dots x_m)=p(x_1)p(x_2)\dots p(x_{m-1})p^{l_{m-1}}(p(x_{m-1})^{-1})=$$

$$p(x_1)p(x_2)\dots p(x_{m-1})p^{l_{m-1}}(p(x_{m-1}^{-1}))=p(x_1)p(x_2)\dots p(x_{m-1})p(p^{l_{m-1}}(x_{m-1}^{-1}))=$$

$$p(x_1)p(x_2)\dots p(x_{m-1})p(x_m),$$

since $x_m = p^{l_{m-1}}(x_{m-1}^{-1})$. This completes the induction for formula (3).

Now, let p be antibalanced. Again, we get $p^k(x^{-1}) = (p^{-k}(x))^{-1}$, for all generators x and integral k. The formula (1) looks for antibalanced permutations as follows:

$$\rho(x_1 x_2 \dots x_m) = p(x_1)p^{-1}(x_2)\dots p^{-1^{(m-1)}}(x_m) \tag{4}$$

(the powers of p alternate between 1 and -1, starting with 1). The proof of formula (4) is very much alike the proof of formula (3) and is left to the reader.

Using formulas (3) and (4) we can proceed to the proof of regularity of our balanced and antibalanced Cayley maps. As we mentioned before, the key point of this proof is the proof of ρ being well-defined. Precisely stated, we want to prove the following: whenever we have two expressions in terms of elements of Ω, $x_1 \dots x_m$ and $y_1 \dots y_k$, such that $x_1 \dots x_m = y_1 \dots y_k$ is a consequence of identities (2), then $\rho(x_1 \dots x_m) = \rho(y_1 \dots y_k)$ is also a consequence of the identities (2). Any consequence of the identities (2) looks as follows:

$$\begin{gathered} x_{1,1}\dots x_{1,n_1}\rho^{i_1}(x_{2,1}\dots x_{2,n})x_{3,1}\dots x_{3,n_2}\rho^{i_2}(x_{4,1}\dots x_{4,n})\dots \\ \rho^{i_k}(x_{2k,1}\dots x_{2k,n})x_{2k+1,1}\dots x_{2k+1,n_{2k+1}} = \\ x_{1,1}\dots x_{1,n_1}\rho^{i'_1}(x_{2,1}\dots x_{2,n})x_{3,1}\dots x_{3,n_2}\rho^{i'_2}(x_{4,1}\dots x_{4,n})\dots \\ \rho^{i'_k}(x_{2k,1}\dots x_{2k,n})x_{2k+1,1}\dots x_{2k+1,n_{2k+1}}, \end{gathered}$$

where $n_1, n_2, \dots, n_{2k+1}, i_1, \dots, i_k, i'_1 \dots, i'_k$ can possibly be zeros and n is the length of equalities in (2). Applying the balanced or antibalanced ρ to both sides of the equation, we get the following equations:

$$\begin{gathered} p(x_{1,1})\dots p(x_{1,n_1})\rho^{i_1+1}(x_{2,1}\dots x_{2,n})p(x_{3,1})\dots p(x_{3,n_2})\rho^{i_2+1}(x_{4,1}\dots x_{4,n})\dots \\ \rho^{i_k+1}(x_{2k,1}\dots x_{2k,n})p(x_{2k+1,1})\dots p(x_{2k+1,n_{2k+1}}) = \\ p(x_{1,1})\dots p(x_{1,n_1})\rho^{i'_1+1}(x_{2,1}\dots x_{2,n})p(x_{3,1})\dots p(x_{3,n_2})\rho^{i'_2+1}(x_{4,1}\dots x_{4,n})\dots \\ \rho^{i'_k+1}(x_{2k,1}\dots x_{2k,n})p(x_{2k+1,1})\dots p(x_{2k+1,n_{2k+1}}), \end{gathered}$$

for a balanced ρ, and

$$\begin{aligned} & p(x_{1,1})\dots p^{\pm 1}(x_{1,n_1})\rho^{i_1+1}(x_{2,1}\dots x_{2,n})p^{\pm 1}(x_{3,1})\dots \\ & \qquad p^{\pm 1}(x_{3,n_2})\rho^{i_2+1}(x_{4,1}\dots x_{4,n})\dots \\ & \qquad \rho^{i_k+1}(x_{2k,1}\dots x_{2k,n})p^{\pm 1}(x_{2k+1,1})\dots p^{\pm 1}(x_{2k+1,n_{2k+1}}) = \\ & p(x_{1,1})\dots p^{\pm 1}(x_{1,n_1})\rho^{i'_1+1}(x_{2,1}\dots x_{2,n})p^{\pm 1}(x_{3,1})\dots \\ & \qquad p^{\pm 1}(x_{3,n_2})\rho^{i'_2+1}(x_{4,1}\dots x_{4,n})\dots \\ & \qquad \rho^{i'_k+1}(x_{2k,1}\dots x_{2k,n})p^{\pm 1}(x_{2k+1,1})\dots p^{\pm 1}(x_{2k+1,n_{2k+1}}), \end{aligned}$$

for an antibalanced ρ, where the signs over the permutation p depend on the parity of the length of the word preceding the particular position, but they are definitely the same on both sides of the equation.

The obtained equations are again consequences of the set of identities (2), for both the balanced and antibalanced case, therefore we conclude that ρ is well defined. Being well-defined and obviously surjective on a finite set, ρ is then a bijection satisfying all the three conditions of a rotary mapping. That completes the proof that our $CM(G_n(\Omega,\chi,p),\Omega,p)$'s are regular. ∎

The next examples illustrate the introduced construction.

Example 1. Let $\Omega=\{a,b,c\}$ and let $\chi(a)=b$, $\chi(b)=a$ and $\chi(c)=c$. Since this says that b is an inverse of a and c is an involution we can simplify the notation by writing $\Omega=\{x,x^{-1},y\}$. Let p be the antibalanced permutation $(xx^{-1}y)$ and let us choose $n=2$. The equivalence classes of irreducible words of length 2 defined by formula (4) are found by means of the following computations:

$$\begin{aligned}\rho(xy)&=p(x)p^{-1}(y)=x^{-1}x^{-1},\\ \rho^2(xy)&=\rho(x^{-1}x^{-1})=p(x^{-1})p^{-1}(x^{-1})=yx,\\ \rho^3(xy)&=\rho(yx)=p(y)p^{-1}(x)=xy.\end{aligned}$$

Similarly, $\rho(xx)=x^{-1}y$, $\rho^2(xx)=\rho(x^{-1}y)=yx^{-1}$, $\rho^3(xx)=xx$.

The group $G_2(\Omega,\chi)$ is the factorization of $F(\Omega,\chi)$ by the identities $xy=x^{-1}x^{-1}=yx$ and $xx=x^{-1}y=yx^{-1}$. It can be easily checked that $G_2(\Omega,\chi)$ is commutative, possessing an involution y and having order 6. Therefore $G_2(\Omega,\chi)\cong(Z_6,\oplus)$, with the correspondence $x\mapsto 1, x^{-1}\mapsto 5, y\mapsto 3$. Obviously, the antibalanced Cayley map $CM(Z_6,\{1,5,3\},p)$ is regular.

Example 2. This is a general example for all the triples (Ω,id_Ω,p). Notice that $\chi=id_\Omega$ yields the identities $e=x\cdot\chi(x)=x^2$, for all $x\in\Omega$, and therefore all the generators are of degree 2. Furthermore, for all $x\in\Omega$, $p(x^{-1})=p(x)^{-1}$ and p is forced to be balanced (and, in fact, irrelevant). Choosing $n=2$ again, we obtain a relatively simple system of identities (2) that does not imply any further identification of elements of $G_2(\Omega,id_\Omega,p)$. Therefore, $|G_2(\Omega,id_\Omega,p)|=1+|\Omega|+(|\Omega|-1)=2|\Omega|$. This makes the "identification" of our group simple. It is of size $2|\Omega|$ and contains $|\Omega|$ involutions. An obvious group that perfectly fits this description is the dihedral group $D_{|\Omega|}$ of all symmetries of an $|\Omega|$-gon. It is left to the reader to check that, indeed, the Cayley maps resulting from our construction are the regular balanced Cayley maps $CM(D_{|\Omega|},\Omega,p)$, with $\Omega=\{y,yx,yx^2,\dots,yx^{|\Omega|-1}\}$, $p=(y\,yx\,yx^2\dots yx^{|\Omega|-1})$. Here, x is the rotation by $2\pi/|\Omega|$ radians, and y is any of the symmetries with respect to an axis.

Example 3. As pointed out in the discussion preceding Theorem 3., the construction does not always grant the pair (Ω,χ) to be a proper generating set for $G_n(\Omega,\chi,p)$.

This situation is illustrated in the following example (which is due to one of our referees who we are thankful to).

Consider again the triple (Ω, id_Ω, p) from the previous example, with $\Omega = \{a,b,c\}$, but n being equal to 3. As before, $x^2 = 1$, for all $x \in \Omega$, and any cyclic p is forced to be balanced. This time, however, (Ω, id_Ω) fails to be a proper generating set for $G_3(\Omega, \chi, p)$. Moreover, the identities (2) force $G_3(\Omega, \chi, p)$ to be almost trivial: besides of the identities $a^2 = b^2 = c^2 = 1$ induced by the choice of χ, the identities (2) achieve the form

$$a \cdot b \cdot c = b \cdot c \cdot a = c \cdot a \cdot b,\ a \cdot c \cdot b = b \cdot a \cdot c = c \cdot b \cdot a,$$
$$a \cdot b \cdot a = b \cdot c \cdot b = c \cdot a \cdot c,\ a \cdot c \cdot a = b \cdot a \cdot b = c \cdot b \cdot c.$$

It follows that $(aba)c = (cac)c = ca$, and $a(bac) = a(acb) = cb$, therefore $a = b$. Similarly, $a = b = c$. Thus, $\Omega = \{a,b,c\}$ is not a proper generating set for $G_3(\Omega, \chi, p) \cong \mathcal{Z}_2$, and we do not get a Cayley map.

Some remarks on the existence of regular Cayley maps that are neither balanced nor antibalanced: Both the balanced and antibalanced regular Cayley maps have been studied in the papers [7], [8]. For quite some time, there have been no examples known of regular Cayley maps that are neither balanced nor antibalanced. As it is argued in the discussion following Theorem 2, the introduced construction has no restrictions on the permutation used but being cyclic. Neither does Theorem 3 rule out the possibility of a regular $CM(G_n(\Omega, \chi, p), \Omega, p)$ based on a permutation p that is not balanced or antibalanced. With the help of the program package GAP we have therefore constructed the groups $G_2(\Omega, \chi, p)$, for all possible triples (Ω, χ, p) with $|\Omega| \leq 6$. Hoping to find some new classes of regular Cayley maps, we have paid particular attention to those supported by permutations neither balanced nor antibalanced. To our surprise (disappointment?), all the $G_2(\Omega, \chi, p)$'s, $|\Omega| \leq 6$, supported by permutations that were neither balanced nor antibalanced, have collapsed into the trivial or 2-element groups (i. e., the identities (2) based on "non-balanced" permutations force too many corollaries identifying all but one or two vertices of the resulting Cayley graph). Since our construction seems to be quite general, this has served to us as a further evidence of the non-existence of regular Cayley maps that are not balanced or antibalanced.

This is, however, not the case. A recent paper of Richter and Širáň [6] provides us with the first example of a regular Cayley map that is neither balanced nor antibalanced, namely, a regular embedding of the icosahedron $C(\mathcal{A}_4, \{(1,2)(3,4), (2,3,4), (2,4,3), (1,2,3), (1,3,2)\})$. Moreover, using techniques developed in their paper, we were able to show the existence of regular Cayley maps for any given cyclic permutation p ([5]). It is therefore quite possible that the technique developed in this paper will eventually produce some examples of regular Cayley maps that are neither balanced nor antibalanced. Let us also mention, that yet another technique for constructing regular Cayley maps is presented in [4].

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